GROUP THEORY 2024 - 25, SOLUTION SHEET 13

Exercise 1. To do yourself. Ask the assistant if something is unclear.

Exercise 2. (1) For each k-vector space V there exists a unique representation $G \curvearrowright V$, since there exists a unique action $G \to GL(V)$.

(2) Let $G \curvearrowright V$ be a one-dimensional representation. Since V doesn't have any proper subspace, it can't have any proper subrepresentation.

(3) The trivial action of G on $\mathbb{C}$ is irreducible by part 2).

Exercise 3. Suppose that V is irreducible. Since $\langle G \cdot v \rangle_{\mathbb{C}} \subseteq V$ is a subrepresentation, it must either be trivial, or V itself. It can't be trivial since $0 \neq v \in \langle G \cdot v \rangle_{\mathbb{C}}$, hence we must have $\langle G \cdot v \rangle_{\mathbb{C}} = V$.

Conversely if V is not irreducible, there exists a proper subrepresentation $W \subset V$. For any non zero $w \in W$ we have that $\{0\} \neq \langle G \cdot w \rangle_{\mathbb{C}} \subseteq W \subsetneq V$.

Exercise 4. (1) Let V be the $\mathbb{C}$-vector space with basis G (as a set). In particular $\dim V = |G|$. Consider the representation $G \curvearrowright V$:

$$g \in G \mapsto \Phi_g : V \to V$$

given on basis elements by $\Phi_g(h) = gh$ for all $h \in G$. Since Φ_g is a bijection on the basis G, it is a linear automorphism as desired. It is immediate to check that this indeed yields a representation of V. Moreover $\Phi_g = \mathrm{Id}_V$ if and only if $g = 1 \in G$ is the identity element, which shows that $G \to GL(V) \cong GL_{|G|}(\mathbb{C})$ is injective.

This representation is called the permutation representation of G.

(2) Fix some non zero vector $v \in V$. It suffices to notice that the set $\{g \cdot v | g \in G\} \subset V$ generates the vector space $\langle G \cdot v \rangle_{\mathbb{C}} = V$ to conclude that

$$\dim V = \dim \langle G \cdot v \rangle_{\mathbb{C}} \leq |\{g \cdot v | g \in G\}| \leq |G|.$$

Exercise 5. (1) Let $\rho : S_n \to GL(\mathbb{C}) \cong \mathbb{C}^\times$ be a one-dimensional representation of S_n. If τ is a transposition, then $id = \rho(id) = \rho(\tau^2) = \rho(\tau)^2$, hence $\rho(\tau) = \pm 1$. As $\mathbb{C}^\times$ is a commutative group, $\rho(ghg^{-1}) = \rho(h)$ for all $g, h \in S_n$. In particular, as all transpositions are conjugate, they are all sent to either 1 or -1. All elements of S_n can be written as a product of transpositions (odd or even number depending on their signature), so we can easily see that if all transpositions are sent to 1, then ρ must be the trivial representation and if they are all sent to -1, then ρ must be the sign representation, i.e. $\rho = sgn$.

(2) (a) A basis for V is given by $\{e_1 - e_n, e_2 - e_n, \ldots, e_{n-1} - e_n\}$. Indeed, all of these elements are contained in V. If $(x_1, \ldots, x_n) \in V$, then $x_n = -\sum_{i<n} x_i$, so we can

write $(x_1, \ldots, x_n) = x_1(e_1 - e_n) + x_2(e_2 - e_n) + \ldots + x_{n-1}(e_{n-1} - e_n)$, hence the family is generating. One can check easily that it is free in $\mathbb{C}^n$, so it is a basis.

(b) Let $\rho : S_n \to GL(V)$ be the representation defined by

$$\rho(\sigma)(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

It is clear that $(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \in V$, as we have just permuted the same coefficients, hence we are done (check by yourself that it is indeed a group morphism!) .

(c) We are going to use exercise 3. Pick $v = (v_1, \ldots, v_n) \in V$. Suppose wlog $v_1 \neq v_2$. Observe that $(12)\cdot v - v = (v_2 - v_1)(e_1 - e_2)$. Therefore, $e_1 - e_2 \in \langle S_n \cdot v\rangle_{\mathbb{C}}$. By acting by suitable permutations on $e_1 - e_2$ we can get all elements of the above-basis, so we conclude that $\langle S_n \cdot v\rangle_{\mathbb{C}} = V$.

Exercise 6. (1) Pick $g, h \in G$ and $T \in Hom_{\mathbb{C}}(V, W)$ arbitrary. Then

$$((gh) \cdot T)(v) = (gh) \cdot (T((gh)^{-1} \cdot v))$$

On the other hand

$$g \cdot (h \cdot T)(v) = g \cdot ((h \cdot T)(h^{-1}v)) = g \cdot ((h \cdot T)(h^{-1}\cdot)(g^{-1}v)) = (gh)T(h^{-1}g^{-1}v)$$

so we are done.

(2) Let us proceed by double inclusion. Pick $T \in Hom_{\mathbb{C}[G]}(V, W)$ arbitrary. By definition, we have that T commutes with the action of any element of G, so $(g\cdot T)(v) = g\cdot T(g^{-1}v) = g \cdot g^{-1} \cdot T(v) = T(v)$, i.e. $g \cdot T = T$ for all $g \in G$, that is $T \in Hom_{\mathbb{C}}(V, W)^G$.

Now, pick $T \in Hom_{\mathbb{C}}(V, W)^G$ arbitrary. Then $T(g\cdot v) = (g\cdot T)(g\cdot v) = g\cdot(T(g^{-1}gv)) = g \cdot (T(v)) = g \cdot T(v)$. As the choice of $g \in G$ and $v \in V$ was arbitrary, we have $T \in Hom_{\mathbb{C}[G]}(V, W)$ and we are done.

Exercise 7. (1) Let $\rho : G \to GL(V)$ be an irreducible representation of G over $\mathbb{C}$. Because G is abelian, for all elements $g \in G$, $\rho(g) : V \to V$ is an intertwiner, so by Schur's lemma $\rho(g)$ is a scalar operator. In particular, every vector subspace of V is fixed by action of all elements of G. As V is irreducible, all such fixed subspaces must be trivial, so equal to either 0 or V. Therefore, V must be 1-dimensional.

(2) By the previous point, all irreducible representations of $\mathbb{Z}/n\mathbb{Z}$ must be 1-dimensional. Let us hence look for group homomorphism $\rho : \mathbb{Z}/n\mathbb{Z} \to GL(\mathbb{C}) \cong \mathbb{C}^\times$. Clearly, such a group morphism is completely determined by an element $\rho([1]) \in \mathbb{C}^\times$ such that $\rho([1])^n = 1$. Hence, we have a correspondence between irreducible representations of $\mathbb{Z}/n\mathbb{Z}$ and n-th roots of unity in $\mathbb{C}$.